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Multiple periodic solutions of a class of p-Laplacian

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Abstract

By using the recent generalization of coincidence degree method, the existence of multiple periodic solutions for a class of p-Laplacian is obtained under the existence of strict upper and lower solutions.

Keywords: Periodic solutions; Strict upper and lower solutions; Coincidence degree; p-Laplacian

1. Introduction

In this paper, we discuss the existence of multiple solutions for the following second order differential equation:

$$\big(\phi_p(x')\big)' + f(t, x, x') = 0, \qquad t \in I =: [0, 1], \tag{1.1}$$

$$x(0) = x(1), \qquad x'(0) = x'(1), \tag{1.2}$$

where $\phi_p(u) = |u|^{p-2}u$ and $p > 1$ is a constant, $f(t, x, y) \in C([0, 1] \times \mathbb{R}^2, \mathbb{R})$ is 1-periodic in t.

If $p = 2$, (1.1) reduces to

$$x'' + f(t, x, x') = 0. \tag{1.3}$$

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By applying lower and upper solutions method, coincidence degree method, many authors obtained the existence results of periodic solutions for (1.2)–(1.3); we refer to [1–5, 7–12] and references therein. But for $p \neq 2$, the results of existence of periodic solutions of (1.1)–(1.2) were relatively few, since the traditional lower and upper solutions method as well as the method of coincidence degree for linear operator cannot be applied directly in this case. In this paper, by using a recent result on generalized coincidence degree method developed in [6], some sufficient conditions for the existence of multiple periodic solutions for (1.1)–(1.2) under the existence of strict lower and upper solutions of (1.1)–(1.2) were obtained.

First, we introduce the concept of strict upper and lower solutions of (1.1)–(1.2).

Definition 1. A function $\alpha \in C^2$ is called a strict lower solution of (1.1)–(1.2) if α satisfies (1.2) and

$$\big(\phi_p\big(\alpha'(t)\big)\big)'(t) + f\big(t, \alpha(t), \alpha'(t)\big) > 0, \quad \forall t \in I.$$

Definition 2. A function $\beta \in C^2$ is called a strict upper solution of (1.1)–(1.2) if β satisfies (1.2) and

$$\big(\phi_p\big(\beta'(t)\big)\big)'(t) + f\big(t, \beta(t), \beta'(t)\big) < 0, \quad \forall t \in I.$$

Definition 3. A function $\psi \in C^1(\mathbb{R}^+, \mathbb{R}^+)$ is called a Nagumo function if ψ is positive and satisfies

$$\int_0^\infty \frac{s\,ds}{\psi(s)} = +\infty.$$

The main result of this paper is

Theorem 1. *Assume the conditions on f hold and there exist n $(n \geqslant 2)$ pairs of strict lower and upper solutions $\{\alpha_k(t), \beta_k(t)\}_{k=1}^n$ of (1.1)–(1.2) such that for all $t \in I = [0,1]$,*

$$\alpha_1(t) < \beta_1(t) < \alpha_2(t) < \beta_2(t) < \cdots < \alpha_n(t) < \beta_n(t),$$

and there exists a Nagumo function $\psi \in C^1(\mathbb{R}^+, \mathbb{R}^+)$ and a $x_0 \gg 1$ such that for $i = 1, 2, \ldots, n$, $\forall t \in I$,

$$-x_0 < \alpha_1(t), \qquad x_0 > \beta_n(t),$$

and

$$f\big(t, \alpha_i(t), 0\big) + x_0 + \alpha_i(t) > 0, \qquad f\big(t, \beta_i(t), 0\big) - x_0 + \beta_i(t) < 0.$$

Moreover, let $d_0 \gg 1$ be defined in (2.7) below such that the function $h(x)$ is well defined in $[-x_0, x_0]$ and for all $x \in [-x_0, x_0]$,

$$h(x) > \max\Big\{c_p \max_{t\in I}\big|\alpha_i'(t)\big|^{\frac{p}{2}}, c_p \max_{t\in I}\big|\beta_i'(t)\big|^{\frac{p}{2}},\ t \in I,\ i = 1, 2, \ldots, n\Big\}.$$

For all $t \in I$, $x \in [-x_0, x_0]$, $y \in \mathbb{R}$, *one has*

$$\left|f(t,x,y)\right| < \psi\left(c_p|y|^{\frac{p}{2}}\right),$$

where $c_p = \sqrt{\frac{2(p-1)}{p}}$.

Then problem (1.1)–(1.2) *has at least* $2n-1$ *periodic solutions* $x_1(t), x_2(t), \ldots, x_n(t)$ *and* $x_1^*(t), x_2^*(t), \ldots, x_{n-1}^*(t)$ *such that for all* $t \in I$,

$$\alpha_1(t) < x_1(t) < \beta_1(t), \quad \alpha_2(t) < x_2(t) < \beta_2(t), \quad \ldots, \quad \alpha_n(t) < x_n(t) < \beta_n(t),$$

and

$$\alpha_i(t) < x_i^*(t) < \beta_{i+1}(t), \quad \textit{and}$$
$$\beta_i(t) < \max_{t\in I} x_i^*(t), \qquad \min_{t\in I} x_i^*(t) < \alpha_{i+1}(t), \quad i = 1, 2, \ldots, n-1.$$

2. Continuation theorem and quasi-linear operator

Definition 4 [6]. Let X and Y be two Banach spaces with norms $\|\cdot\|_X$ and $\|\cdot\|_Y$, respectively. A continuous operator

$$M : X \cap \operatorname{dom} M \to Y$$

is said to be quasi-linear if

(I) $\operatorname{Im} M = M(X \cap \operatorname{dom} M)$ is a closed subset of Y,
(II) $\ker M = \{x \in X \cap \operatorname{dom} M\colon\ Mx = 0\}$ is linearly homeomorphic to $\mathbb{R}^n$, $n \in \mathbb{N}$.

Definition 5 [6]. Let $X_1 = \ker M$ and X_2 be the complement space of X_1 in X: $X = X_1 \oplus X_2$. Y_1 is a subspace of Y and Y_2 is the complement of Y_1 in Y: $Y = Y_1 \oplus Y_2$. Let $P : X \to X_1$ and $Q : Y \to Y_1$ be two projectors and $\Omega \subset X$ be an open and bounded set with the origin $\theta \in \Omega$, where θ is the origin of a linear space.

Assume $N_\lambda : \bar{\Omega} \to Y$, $\lambda \in [0,1]$, is a continuous operator and denote N_1 by N. Let $\Sigma_\lambda = \{x \in \bar{\Omega}\colon\ Mx = N_\lambda x\}$. N_λ is called M-compact in $\bar{\Omega}$ if

(III) there is a vector subspace Y_1 of Y with $\dim Y_1 = \dim X_1$ and an operator $R : \bar{\Omega} \times [0,1] \to X_2$ being continuous and compact such that for $\lambda \in [0,1]$,

$$(E-Q)N_\lambda(\bar{\Omega}) \subset \operatorname{Im} M \subset (E-Q)Y, \tag{2.1}$$
$$QN_\lambda x = \theta, \quad \lambda \in (0,1) \quad \Leftrightarrow \quad QNx = \theta, \tag{2.2}$$

where E is the identity operator and

$$R(\cdot,0) \quad \text{is the zero operator and} \quad R(\cdot,\lambda)|_{\Sigma_\lambda} = (E-P)|_{\Sigma_\lambda}, \tag{2.3}$$
$$M\left[P + R(\cdot,\lambda)\right] = (E-Q)N_\lambda. \tag{2.4}$$

Lemma A [6, Lemma 2.1]. *Let $J : Y_1 \to X_1$ be a homeomorphism with $J(\theta) = \theta$. Define $S_\lambda : \bar{\Omega} \cap \operatorname{dom} M \to X$, $0 \leqslant \lambda \leqslant 1$, by*

$$S_\lambda = P + R(\cdot, \lambda) + JQN. \tag{2.5}$$

Then S_λ is a completely continuous mapping and the abstract equation

$$Mx = N_\lambda x, \quad \lambda \in (0, 1], \tag{2.6}$$

has a solution $x \in \bar{\Omega}$ if and only if $x \in \bar{\Omega}$ is a fixed point of S_λ.

Theorem B [6, Theorem 2.1]. *If $M : X \cap \operatorname{dom} M \to Y$ is a quasi-linear operator and $N_\lambda : \bar{\Omega} \to Y$, $\lambda \in [0, 1]$, is M-compact, in addition, if*

(H_1) $Mx \neq N_\lambda x$, $\lambda \in (0, 1)$, $x \in \partial\Omega$,
(H_2) $\deg\{JQN, \Omega \cap \ker M, 0\} \neq 0$,

where $N = N_1$, then the abstract equation $Mx = Nx$ has at least one solution in $\bar{\Omega}$.

Now we define

$$I = [0, 1], \qquad X = \{x \in C^1(I)\colon\ x(0) = x(1),\ x'(0) = x'(1)\}, \qquad Y = C(I),$$
$$M : X \cap \operatorname{Dom} M \to Y, \qquad \operatorname{Dom} M = \{x \in C^1(I)\colon\ \phi_p(x') \in C^1(I)\},$$
$$Mx = (\phi_p(x'))',$$
$$N : X \to Y, \qquad Nx(t) = -f(t, x(t), x'(t)),$$

and

$$X_1 = \ker M = \{c \in \mathbb{R}\colon\ c \text{ is constant}\} = \mathbb{R}, \qquad Y_1 = X_1 = \ker M,$$
$$P : X \to X_1\colon\ Px = x(0), \qquad Q : Y \to Y_1\colon\ Qy = \int_0^1 f(t, y(t), y'(t))\, dt,$$
$$N_\lambda x(t) = -\lambda f(t, x(t), x'(t)).$$

Then

$$\dim X_1 = \dim Y_1 = 1 < \infty.$$

It is easy to see that M is a quasi-linear operator. Now, problem (1.1)–(1.2) is equivalent to the following operator equation:

$$Mx = Nx.$$

We show next the conditions (2.1)–(2.4) hold.

For any $\bar{\Omega} \in X$, it is easy to see that (2.1) and (2.2) hold.

Let

$$R(x, \lambda)(t) = \int_0^t \phi_p^{-1}\left[c - \lambda \int_0^s f(\tau, x(\tau), x'(\tau))\, d\tau\right] ds.$$

If $c=\phi_p(x'(0))$, then for $x\in\Sigma_\lambda$, that is, $Mx=N_\lambda x$, we get

$$\begin{aligned}
R(x,\lambda)(t) &= \int_0^t \phi_p^{-1}\left[\phi_p\big(x'(0)\big)+\int_0^s \phi_p\big(x'(\tau)\big)'\,d\tau\right]ds \\
&= \int_0^t \phi_p^{-1}\big[\phi_p\big(x'(0)\big)+\phi_p\big(x'(s)\big)-\phi_p\big(x'(0)\big)\big]\,ds \\
&= \int_0^t \phi_p^{-1}\big[\phi_p\big(x'(s)\big)\big]\,ds \\
&= \int_0^t x'(s)\,ds = x(t)-x(0)=\big[(E-P)x\big](t),
\end{aligned}$$

which shows that the second part of (2.3) holds.

If $x\in X$, then $R(x,\lambda)(1)=x(1)-x(0)=0$ and for $\lambda=0$ it follows from $(\phi_p(x'(t)))'=0$ that $x'(t)=x'(0)=\text{constant}$, which, together with $x(1)=x(0)$, implies that $c=\phi_p(x'(0))=0$. Therefore

$$R(x,0)(t)=\int_0^t \phi_p^{-1}(c)\,ds=\phi_p^{-1}(c)t=0,$$

which shows that the first part of (2.3) holds. Finally, by using the similar method used in [6], one can verify that (2.4) holds. In fact, both sides of (2.4) equal to $-\lambda f(t,x(t),x'(t))$ if $x\in X$. ($x'(1)=x'(0)\Leftrightarrow\int_0^1 f(t,x(t),x'(t))\,dt=0$.)

The above discussion shows that the operator N_λ is M-compact in $\bar{\Omega}$.

Define $J: Y_1\to X_1: J(c)=c$, that is, $J=E$ is the unit operator; we have

$$S_\lambda=P+R(\cdot,\lambda)+JQN=P+R(\cdot,\lambda)+QN$$

and S_λ is completely continuous.

Let $x_0>0$, $d_0\gg 1$ and $h(x)$ be the solution of the following initial value problem:

$$uu'+\psi(u)=0,\qquad u(-x_0)=d_0, \tag{2.7}$$

where ψ is the Nagumo function defined in Definition 3. Then we have

Lemma 1. *The function $h(x)$ is well defined and positive on $[-x_0,x_0]$. Moreover, if $d_0\gg 1$, then $h(x)\gg 1$ for all $x\in[-x_0,x_0]$.*

Proof. Assume there exists a $x\in(-x_0,x_0]$ such that $h(x)=0$. Let $x^*=\inf\{x\colon h(x)=0,\ x\in(-x_0,x_0]\}$. Rewrite (2.7) as

$$-\frac{uu'}{\psi(u)}=1,\quad u>0.$$

Integrate above equation over $[-x_0, x^*]$ to get

$$-\int_{x_0}^{x^*} \frac{u(s)u'(s)}{\psi(u(s))}\,ds = \int_0^{d_0} \frac{\tau\,d\tau}{\psi(\tau)} = x^* + x_0 \leqslant 2x_0.$$

But the left side of above equation goes to $+\infty$ as d_0 goes to $+\infty$ by Definition 3. This is a contraction. Hence $h(x) > 0$ for $x \in [-x_0, x_0]$, moreover, since $d_0 \gg 1$, by the continuity of solution of differential equations on the initial values, we see that $h(x) \gg 1$ for $x \in [-x_0, x_0]$. □

The following result is well known.

Lemma 2. *For any interval* $[a, b] \subset \mathbb{R}$, *let* H *be any one-dimensional continuous mapping* $H : R \to R$. *If* $H(a) > 0$ *and* $H(b) < 0$, *then* $\mathrm{Deg}_B(H, (a, b), 0) = -1$, *if* $H(a) < 0$ *and* $H(b) > 0$, *then* $\mathrm{Deg}_B(H, (a, b), 0) = 1$, *where* Deg_B *denotes the Brouwer degree.*

Define the set

$$\Omega = \bigl\{x \in X\colon\ \bigl|x(t)\bigr| < x_0,\ 2(p-1)\bigl|x'(t)\bigr|^p < ph^2\bigl(x(t)\bigr),\ \forall t \in I\bigr\}. \tag{2.8}$$

Then we have

Lemma 3. *Let* x_0 *be given in* (2.7) *and* Deg *be the coincidence degree. If the following conditions hold*:

(a) $f(t, -x_0, 0) > 0$, $f(t, x_0, 0) < 0$, $\forall t \in I$;
(b) $|f(t, x, y)| < \psi(c_p|y|^{\frac{p}{2}})$, $\forall t \in I$, $|x| \leqslant x_0$, $y \in \mathbb{R}$, *where* $c_p = \sqrt{\frac{2(p-1)}{p}}$,

then

$$\mathrm{Deg}\bigl[(M, N), \Omega\bigr] = -1.$$

Proof. Consider the following family of equations:

$$Mx = \lambda Nx, \quad \lambda \in (0, 1]. \tag{2.9}$$

We shall show

$$Mx \neq \lambda Nx, \quad \forall x \in \partial\Omega,\ \lambda \in (0, 1]. \tag{2.10}$$

If not, then there exists some $\lambda \in (0, 1]$ and $x \in \partial\Omega$, such that (2.9) holds. Then there exists $t_0 \in [0, 1]$ such that either $|x(t_0)| = x_0$ or $2(p-1)|x'(t_0)|^p = ph^2(x(t_0))$. We have therefore two possibilities:

Case (i). $t_0 = 0$ or $t_0 = 1$. In this case, $|x(0)| = x_0 = |x(1)|$, or $2(p-1)|x'(0)|^p = 2(p-1)|x'(1)|^p = ph^2(x(0))$. If $x(0) = x(1) = x_0$, then $x(0) = x(1) = \max_{t\in[0,1]} x(t)$, in this case we have $x'(0) \leqslant 0$, $x'(1) \geqslant 0$. But by $x'(0) = x'(1)$, we have $x'(0) = x'(1) = 0$, $x''(0) \leqslant 0$. If we define

$$\lim_{u \to 0} |u|^{p-2} = \begin{cases} 0, & \text{if } p > 2, \\ 1, & \text{if } p = 2, \\ +\infty, & \text{if } p < 2, \end{cases}$$

then

$$\lim_{x'(0) \to 0} (p-1)\left|x'(0)\right|^{p-2} x''(0) \leqslant 0,$$

but

$$\lim_{x'(0) \to 0} (p-1)\left|x'(0)\right|^{p-2} x''(0) = -\lambda f\bigl(0, x(0), x'(0)\bigr) = -\lambda f(0, x_0, 0) > 0,$$

which is a contraction. Similarly, we will get a contradiction if $x(0) = x(1) = -x_0$.

Now we assume $2(p-1)|x'(0)|^p = ph^2(x(0))$, $2(p-1)|x'(1)|^p = ph^2(x(1))$. In this case, we consider the function

$$g(t) = 2(p-1)\left|x'(t)\right|^p - ph^2\bigl(x(t)\bigr).$$

Then $g(t) \leqslant 0$, $g(0) = g(1) = 0$, hence $g'(0) \leqslant 0$, $g'(1) \geqslant 0$. But on the other hand,

$$\begin{aligned} g'(t) &= 2px'(t)\bigl[(p-1)\left|x'(t)\right|^{p-2} x''(t) - h\bigl(x(t)\bigr)h'\bigl(x(t)\bigr)\bigr] \\ &= 2px'(t)\bigl[(p-1)\left|x'(t)\right|^{p-2} x''(t) + \psi\bigl(h\bigl(x(t)\bigr)\bigr)\bigr], \end{aligned}$$

and this implies that

$$\begin{aligned} g'(0) &= 2px'(0)\bigl[-\lambda f\bigl(0, x(0), x'(0)\bigr) + \psi\bigl(c_p\left|x'(0)\right|^{\frac{p}{2}}\bigr)\bigr], \\ g'(1) &= 2px'(1)\bigl[-\lambda f\bigl(0, x(1), x'(1)\bigr) + \psi\bigl(c_p\left|x'(1)\right|^{\frac{p}{2}}\bigr)\bigr], \end{aligned}$$

it follows from (b) that $g'(0)g'(1) > 0$. A contradiction!

Case (ii). $t_0 \in (0,1)$. If $x(t_0) = x_0$, then $x(t_0)$ is a maximum value, hence $x'(t_0) = 0$, $x''(t_0) \leqslant 0$. But it follows from (a) and above discussion that

$$\lim_{t \to t_0} (p-1)\left|x'(t)\right|^{p-2} x''(t) = -\lambda f\bigl(t_0, x(t_0), x'(t_0)\bigr) = -\lambda f(t_0, x_0, 0) > 0,$$

which is a contradiction. Similarly, we shall get a contraction if $x(t_0) = -x_0$.

Now suppose $2(p-1)|x'(t_0)|^p = ph^2(x(t_0))$. Then there are two possibilities:

(A) $x'(t_0) > 0$;
(B) $x'(t_0) < 0$.

In case (A), let $g(t)$ be defined in above, then $g(t) \leqslant 0$, $\forall t \in [0,1]$, and $g(t_0) = \max_{0 \leqslant t \leqslant 1} g(t) = 0$, which yields $g'(t_0) = 0$. But in this case, we have

$$g'(t_0) = 2px'(t_0)\bigl[-\lambda f\bigl(t_0, x(t_0), x'(t_0)\bigr) + \psi\bigl(c_p\left|x'(t_0)\right|^{\frac{p}{2}}\bigr)\bigr] > 0.$$

In case (B), we have

$$g'(t_0) = 2px'(t_0)\left[-\lambda f\left(t_0, x(t_0), x'(t_0)\right) + \psi\left(c_p\left|x'(t_0)\right|^{\frac{p}{2}}\right)\right] < 0.$$

In both cases, we get $g'(t_0) \neq 0$, which contradicts the assumption $g'(t_0) = 0$. Combining the results of (i) and (ii), we obtain (2.10).

On the other hand, for $\lambda \in (0,1]$, it follows from [6, Lemma 1] that (2.9) is equivalent to the following family of operator equations:

$$x = Px + R(x,\lambda) + JQNx = Px + R(x,\lambda) + QNx. \tag{2.11}$$

Define a mapping $\bar{H} : \bar{\Omega} \times [0,1] \to X$,

$$\bar{H}(x,\lambda) = Px + R(x,\lambda) + QNx,$$

then it is easy to see that $\bar{H}$ is completely continuous and we claim that

$$x \neq \bar{H}(x,\lambda), \quad \forall x \in \partial\Omega,\ \lambda \in [0,1]. \tag{2.12}$$

In fact, for $\lambda \in (0,1]$, it follows from (2.10), (2.11) that (2.12) holds. For $\lambda = 0$, if there exists a $x \in \partial\Omega$, such that $x = \bar{H}(x,0)$, that is $x = Px + QNx$, in this case, $QNx = 0$, $x \in \ker M$, hence $x = x_0$ or $x = -x_0$. But

$$QN(x_0) = -\int_0^1 f(t, x_0, 0)\,dt > 0, \qquad QN(-x_0) = -\int_0^1 f(t, -x_0, 0)\,dt < 0,$$

but this contradicts $x = Px + QNx$ for $x \in \ker M$. Therefore (2.12) holds. It follows from [4,6] and by using the invariance of Leray–Schauder degree under homotopy, we obtain

$$\begin{aligned}
\operatorname{Deg}\left[(M,N),\Omega\right] &= \operatorname{Deg}\left(I - \bar{H}(\cdot,1), \Omega, 0\right) \\
&= \operatorname{Deg}\left(I - \bar{H}(\cdot,0), \Omega, 0\right) \\
&= \operatorname{Deg}(I - P - QN, \Omega, 0) \\
&= \operatorname{Deg}_B\left((I - P - QN)|_{\ker M \cap \bar{\Omega}}, \ker M \cap \Omega, 0\right) \\
&= \operatorname{Deg}_B\left((-QN)|_{\ker M \cap \bar{\Omega}}, \ker M \cap \Omega, 0\right).
\end{aligned}$$

Since $\ker M$ is one-dimensional and $QN(x_0) > 0$, $QN(-x_0) < 0$, we get from Lemma 2

$$\operatorname{Deg}_B\left((-QN)|_{\ker M \cap \bar{\Omega}}, \ker M \cap \Omega, 0\right) = -1. \qquad \square$$

Lemma 4. *Assume there exist lower and upper solutions $\alpha(t), \beta(t)$ of (1.1)–(1.2) respectively with $\alpha(t) < \beta(t)$, $t \in I$, and a Nagumo function $\psi \in C^1[0,\infty)$ such that for $x, y \in R$, $x \in (\alpha(t), \beta(t)) \cup (-\beta(t), -\alpha(t))$, $t \in I$,*

$$\left|f(t,x,y)\right| < \psi\left(c_p|y|^{\frac{p}{2}}\right),$$

and choose $x_0 > 0$ large enough such that for all $t \in I$,

$$x_0 > \beta(t) \quad \textit{and} \quad -x_0 < \alpha(t),$$
$$f\left(t,\alpha(t),0\right) + x_0 + \alpha(t) > 0 \quad \textit{and} \quad f\left(t,\beta(t),0\right) - x_0 + \beta(t) < 0.$$

We choose also $d_0 > 0$ large enough, such that for all $x \in [-x_0, x_0]$,

$$h(x) > \max\Big\{\max_{t\in I}\big|\alpha'(t)\big|, \max_{t\in I}\big|\beta'(t)\big|\Big\}.$$

Define a set $\Omega_{\alpha,\beta}$ as

$$\Omega_{\alpha,\beta} = \big\{x \in C^1(I)\colon\ \alpha(t) < x < \beta(t),\ 2(p-1)\big|x'(t)\big|^p < ph^2\big(x(t)\big),\ \forall t \in I\big\}.$$

Then

$$\operatorname{Deg}\big[(M, N), \Omega_{\alpha,\beta}\big] = -1.$$

Proof. Define the functions f^*, F as

$$f^*(t,x,y) = \begin{cases} f(t,x,h(x)), & y > d_p h^{\frac{2}{p}}(x),\ |x| \leqslant x_0, \\ f(t,x,y), & |y| \leqslant d_p h^{\frac{2}{p}}(x),\ |x| \leqslant x_0, \\ f(t,x,-h(x)), & y < -d_p h^{\frac{2}{p}}(x),\ |x| \leqslant x_0, \end{cases}$$

where

$$d_p = \left(\frac{p}{2(p-1)}\right)^{\frac{1}{p}},$$

and

$$F(t,x,y) = \begin{cases} f^*(t,\beta(t),y) + \beta(t) - x, & \beta(t) < x \leqslant x_0, \\ f^*(t,x,y), & \alpha(t) \leqslant x \leqslant \beta(t), \\ f^*(t,\alpha(t),y) + \alpha(t) - x, & -x_0 \leqslant x < \alpha(t), \end{cases}$$

and we then extend F to $I \times R^2$ and 1-periodic in t as a bounded and continuous function. Now for all $t \in I$,

$$F(t,-x_0,0) = f^*\big(t,\alpha(t),0\big) + \alpha(t) + x_0 = f\big(t,\alpha(t),0\big) + \alpha(t) + x_0 > 0,$$
$$F(t,x_0,0) = f^*\big(t,\beta(t),0\big) + \beta(t) - x_0 = f\big(t,\beta(t),0\big) + \beta(t) - x_0 < 0,$$

it follows from Lemma 2 that,

$$\operatorname{Deg}\big[(M, N_F), \Omega\big] = -1,$$

where

$$N_F x(t) = -F\big(t, x(t), x'(t)\big), \quad \forall t \in I.$$

Next we show

$$\operatorname{Deg}\big[(M, N_F), \Omega_{\alpha,\beta}\big] = -1. \tag{2.13}$$

It suffices to show

$$Mx \neq N_F x, \quad \forall x \in \bar{\Omega} \setminus \Omega_{\alpha,\beta}. \tag{2.14}$$

In fact, let $x \in \bar{\Omega}$ such that $Mx = N_F x$ and assume

$$\max_{t\in I}\big\{x(t) - \beta(t)\big\} = x(t_0) - \beta(t_0) > 0.$$

Then the function $x(t)-\beta(t)$ attains its maximum value on $t_0 \in [0,1]$. There are two possibilities: (I) $t_0=0$. Then $x'(0)-\beta'(0)\leqslant 0$. By periodicity of x and β, we have

$$x(0)-\beta(0)=x(1)-\beta(1),$$

hence $x'(1)-\beta'(1)\geqslant 0$. Again by periodicity, we get

$$x'(0)-\beta'(0)=x'(1)-\beta'(1)=0.$$

(II) $t_0\in(0,1)$. Then we have $x(t_0)-\beta(t_0)>0$ and $x'(t_0)-\beta'(t_0)=0$. In all above cases, we have

$$x'(t_0)=\beta'(t_0), \qquad x''(t_0)-\beta''(t_0)\leqslant 0,$$

which implies that

$$\begin{aligned}&(p-1)\left|x'(t_0)\right|^{p-2}x''(t_0)-(p-1)\left|\beta'(t_0)\right|^{p-2}\beta''(t_0)\\&\quad=(p-1)\left|\beta'(t_0)\right|^{p-2}\bigl(x''(t_0)-\beta''(t_0)\bigr)\leqslant 0.\end{aligned}$$

But on the other hand,

$$\begin{aligned}&(p-1)\left|x'(t_0)\right|^{p-2}x''(t_0)-(p-1)\left|\beta'(t_0)\right|^{p-2}\beta''(t_0)\\&\quad=-F\bigl(t_0,x(t_0),x'(t_0)\bigr)-\bigl(\phi_p\bigl(\beta'(t)\bigr)\bigr)'\big|_{t=t_0}\\&\quad=-f\bigl(t_0,\beta(t_0),\beta'(t_0)\bigr)-\bigl(\phi_p\bigl(\beta'(t)\bigr)\bigr)'\big|_{t=t_0}+x(t_0)-\beta(t_0)\\&\quad>0,\end{aligned}$$

this is a contradiction. Therefore we have $x(t)\leqslant\beta(t)$, $\forall t\in I$. Similarly, we can prove $\alpha(t)\leqslant x(t)$, $\forall t\in I$.

Now we assume

$$\begin{aligned}&\max_{t\in I}\bigl\{m(t)=:2(p-1)\left|x'(t)\right|^{p}-ph^2\bigl(x(t)\bigr)\bigr\}\\&\quad=m(t_0)=2(p-1)\left|x'(t_0)\right|^{p}-ph^2\bigl(x(t_0)\bigr)>0.\end{aligned}$$

Then we have $x'(t_0)\neq 0$ and

$$m'(t_0)=\begin{cases}\leqslant 0, & t_0=0;\\ 0, & t_0\in(0,1);\\ \geqslant 0, & t_0=1.\end{cases}\tag{2.15}$$

But on the other hand, if $t_0\in(0,1)$, we have

$$\begin{aligned}m'(t_0)&=2p\bigl[(p-1)\left|x'(t_0)\right|^{p-2}x''(t_0)-h\bigl(x(t_0)\bigr)h'\bigl(x(t_0)\bigr)\bigr]\\&=2px'(t_0)\bigl[-F\bigl(t_0,x(t_0),x'(t_0)\bigr)+\psi\bigl(h\bigl(x(t_0)\bigr)\bigr)\bigr]\\&=2px'(t_0)\bigl[-f\bigl(t_0,x(t_0),\pm d_ph^{\frac{2}{p}}\bigl(x(t_0)\bigr)\bigr)+\psi\bigl(h\bigl(x(t_0)\bigr)\bigr)\bigr],\\&\neq 0,\end{aligned}$$

which is a contradiction.

Similarly, as $x'(0)=x'(1)$, we get $m'(0)m'(1)>0$. But this contradicts (2.15), which yields $m'(0)m'(1)\leqslant 0$. Hence $2(p-1)|x'(t)|^p\leqslant ph^2(x(t))$, $\forall t\in I$. Combining above results, we see that if $x \in \bar{\Omega}$, $Mx = N_F x$, then $x \in \Omega_{\alpha,\beta}$. Hence (2.14) is proved. Since in $\Omega_{\alpha,\beta}$, $F = f$, we have $N = N_F$. From the property of coincidence, we know that (2.13) holds. Lemma 4 is thus proved. □

3. Proof of Theorem 1

We select $x_0 > 0$ large enough such that for $i = 1, 2, \dots, n$, $\forall t \in I$,

$$-x_0 < \alpha_1(t), \qquad x_0 > \beta_n(t),$$

and

$$f\big(t, \alpha_i(t), 0\big) + x_0 + \alpha_i(t) > 0, \qquad f\big(t, \beta_i(t), 0\big) - x_0 + \beta_i(t) < 0.$$

Moreover, let $d_0 \gg 1$ such that the function $h(x)$ is well defined in $[-x_0, x_0]$ and for all $x \in [-x_0, x_0]$,

$$h(x) > \max\Big\{c_p \max_{t\in I}\big|\alpha_i'(t)\big|^{\frac{p}{2}},\ c_p \max_{t\in I}\big|\beta_i'(t)\big|^{\frac{p}{2}},\ t \in I,\ i = 1, 2, \dots, n\Big\}.$$

For $i = 1, 2, \dots, n$, define the set Ω_i as

$$\Omega_i = \big\{x \in X\colon\ \alpha_i(t) < x(t) < \beta_i(t),\ 2(p-1)\big|x'(t)\big|^p < ph^2\big(x(t)\big),\ \forall t \in I\big\},$$

and define the set Ω_{n+1} as

$$\Omega_{n+1} = \big\{x \in X\colon\ \alpha_1(t) < x(t) < \beta_n(t),\ 2(p-1)\big|x'(t)\big|^p < ph^2\big(x(t)\big),\ \forall t \in I\big\}.$$

Then it follows from Lemma 3 that

$$\operatorname{Deg}\big[(M, N), \Omega_i\big] = -1, \quad i = 1, 2, \dots, n+1.$$

From the additive property of coincidence degree, we obtain

$$\operatorname{Deg}\big[(M, N), \Omega_{n+1} \setminus \overline{\Omega_1 \cup \Omega_2 \cup \cdots \cup \Omega_n}\big] = n - 1 \geqslant 1.$$

Since $n \geqslant 2$ is arbitrary, we first consider the case $n = 2$. The above discussion implies that the equation $Mx = Nx$, that is, the equation (1.1)–(1.2) has at least one solution in the set Ω_1, Ω_2 and $\Omega_3 \setminus \overline{\Omega_1 \cup \Omega_2}$, respectively. That is, there exist periodic solutions $x_1(t), x_2(t), x_1^*(t)$ such that

$$\alpha_1(t) < x_1(t) < \beta_1(t), \qquad \alpha_2(t) < x_2(t) < \beta_2(t)$$

and

$$\alpha_1(t) < x_1^*(t) < \beta_2(t), \quad \text{and} \quad \beta_1(t) < \max_{t\in I} x_1^*(t), \qquad \min_{t\in I} x_1^*(t) < \alpha_2(t).$$

For $n = 3$, replacing $\alpha_1, \beta_1, \alpha_2, \beta_2$ by $\alpha_2, \beta_2, \alpha_3, \beta_3$ respectively, we obtain another two periodic solutions $x_3(t), x_2^*(t)$ of (1.1)–(1.2) such that

$$\alpha_3(t) < x_3(t) < \beta_3(t),$$

and

$$\alpha_2(t) < x_2^*(t) < \beta_3(t), \quad \text{and} \quad \beta_2(t) < \max_{t\in I} x_2^*(t), \qquad \min_{t\in I} x_2^*(t) < \alpha_3(t).$$

In this way, we obtain by induction the existence of periodic solutions

$$x_1(t),\ x_2(t),\ \ldots,\ x_n(t),\ x_1^*(t),\ x_2^*(t),\ \ldots,\ x_{n-1}^*(t)$$

of (1.1)–(1.2) such that

$$\alpha_i(t) < x_i(t) < \beta_i(t), \quad i = 1, 2, \ldots, n,$$

and

$$\alpha_i(t) < x_i^*(t) < \beta_{i+1}(t), \quad \text{and}$$
$$\beta_i(t) < \max_{t\in I} x_i^*(t), \qquad \min_{t\in I} x_i^*(t) < \alpha_{i+1}(t), \quad i = 1, 2, \ldots, n-1.$$

This completes the proof of Theorem 1. □

Example. Consider Liénard equation

$$x'' + f(x)x' + g(x) = h(t), \tag{3.1}$$
$$x(0) = x(1), \qquad x'(0) = x'(1), \tag{3.2}$$

where f, g satisfy local Lipschitz conditions, $h \in C(I)$ is 1-periodic. If there exist constants c_i, d_i, $i = 1, 2, \ldots, m$, with $m \geqslant 2$ such that

$$c_1 < d_1 < c_2 < d_2 < \cdots < c_m < d_m,$$
$$g(c_i) > h_0, \qquad g(d_i) < h_0, \quad i = 1, 2, \ldots, m,$$

where $h_0 = \max_{t\in I} |h(t)|$, then (3.1)–(3.2) has at least $2m - 1$ periodic solutions $x_1(t)$, $x_2(t), \ldots, x_m(t), x_1^*(t), x_2^*(t), \ldots, x_{m-1}^*(t)$ such that

$$c_i < x_i(t) < d_i, \quad i = 1, 2, \ldots, m,$$

and

$$c_i < x_i^*(t) < d_{i+1}, \quad \text{and}$$
$$d_i < \max_{t\in I} x_i^*(t), \qquad \min_{t\in I} x_i^*(t) < c_{i+1}, \quad i = 1, 2, \ldots, m-1.$$

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